

# IDENTIFICATION OF BESOV SPACES VIA LITTLEWOOD-PALEY-STEIN TYPE $g$ -FUNCTIONS

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**ABSTRACT.** We use Littlewood-Paley-Stein type  $g$ -functions (also called generalized square functions) associated to symmetric diffusion semigroups to obtain a characterization of inhomogeneous abstract Besov spaces on the abstract Hilbert spaces. Then we apply our results for the abstract Besov spaces defined through the Poisson and Gauss-Weierstrass semigroups.

## 1. INTRODUCTION

Let  $\{T_t\}_{t \geq 0}$  be a symmetric diffusion semigroup on a Hilbert space  $\mathcal{H}$ . (See Section 3 for the definition.). For  $1 \leq q < \infty$ ,  $\alpha > 0$ , the *inhomogeneous abstract Besov space*  $B_q^\alpha := B_q^\alpha(\mathcal{H})$  is the space of all functions  $f \in \mathcal{H}$  for which

$$\int_0^1 (s^{-\alpha} \Omega_r(s, f))^q \frac{ds}{s} < \infty.$$

Here,  $\Omega_r(s, f)$  is the *modulus of continuity* for  $r \in \mathbb{N}$  with  $r \geq \alpha$  and is defined by

$$\Omega_r(s, f) = \sup_{0 < \tau \leq s} \|(I - T_\tau)^r f\|.$$

The space  $B_q^\alpha$ ,  $1 \leq q < \infty$ , is a Banach space with the norm

$$\|f\|_{B_q^\alpha} = \|f\| + \left( \int_0^1 (s^{-\alpha} \Omega_r(s, f))^q \frac{ds}{s} \right)^{1/q}. \quad (1)$$

We use the standard convention for the definition of the norm when  $q = \infty$ . Notice that the Besov norm is independent of the choice of  $r$  due to the monotonicity of the modulus of continuity  $\Omega_r$ . Therefore, for fixed  $q$  and  $\alpha$ , the definition of Besov space is independent of the choice of  $r$ .

In our previous paper [22], we proved the norm in (1) can be expressed in terms of “smooth” and “band-limited” functions using the Littlewood-Paley decomposition. In this paper we show this norm is also equivalent to a norm which can be expressed in terms of Littlewood-Paley-Stein type  $g$ -functions as we will explain.

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For any given  $m \in \mathbb{N}$ ,  $1 \leq q < \infty$ ,  $\beta \in \mathbb{R}$  and a diffusion semigroup  $\{T_t\}_{t \geq 0}$  on a Hilbert space  $\mathcal{H}$ , we denote a bounded map from  $\mathcal{H}$  to  $\mathcal{H}$  by  $\mathcal{G}_{m,q,\beta}$  and call it a *Littlewood-Paley-Stein type  $g$ -function* if the following property holds:

$$\|\mathcal{G}_{m,q,\beta}(f)\|^q \asymp \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m}{\partial t^m} T_t f \right\|^q \frac{dt}{t}. \quad (2)$$

For  $q = \infty$  we use the standard convention.

The functions  $\mathcal{G}_{m,q,\beta}$  can be interpreted as a generalized version of  *$g$ -functions* or *square functions* in abstract form. In the classical Littlewood-Paley theory, various  $g$ -functions have played a central role. Typical examples are

$$g_k(f)(x) = \left( \int_0^\infty |t^k \frac{\partial^k}{\partial t^k} T_t f(x)|^2 \frac{dt}{t} \right)^{1/2} \quad (3)$$

and

$$g_k(f)(x) = \left( \int_0^\infty |(tA)^k \frac{\partial^k}{\partial t^k} T_t f(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad (4)$$

where  $f \in L^p$ ,  $A$  is the infinitesimal operator of the semigroup and  $k \in \mathbb{N}$ . (Note, for  $p = 2$ , if we take the  $L^2$ -norm of (3), we arrive at (2) with  $q = 2$ ,  $\beta = 0$  and  $m = k$ . Therefore, our definition of a Littlewood-Paley-Stein type  $g$ -function on an abstract Hilbert space as a generalized version of a  $g$ -function makes intuitive sense.)

The  $g$ -functions for the generators of semigroups grew out of classical harmonic analysis and the Littlewood-Paley theory. They were first developed in E. M. Stein's classical book [27]. Also see the monograph [13] for when  $k = 1$  and [23] for any  $k \in \mathbb{N}$ . As it has been shown in [13], the  $g$ -functions associated to the Poisson semigroup can be applied to determine the bounded solutions of a Dirichlet problem on  $\mathbb{R}^n \times \mathbb{R}^+$  with bounded boundary value function. Note in his work [23], the author proves the  $L^p$ -boundedness of  $g$ -functions. These functions here are associated to  $(tA)^s T_t$ , where  $s > 0$  and  $A$  is the infinitesimal operator for the semigroup  $\{T_t\}_{t \geq 0}$ . Another principal motivation for introducing the  $g$ -functions is to provide new equivalent norms for the spaces  $L^p$ , which make the boundedness properties of various operators derived from the semigroup  $T_t$  more transparent. For further applications of this form, for example in unconditionally martingale difference spaces, we refer the reader to [20], where  $\mathcal{H} = \mathbb{R}^n$ , and  $\beta = 0$ .

The main result of this paper, which characterizes the inhomogenous abstract Besov spaces in terms of the Littlewood-Paley-Stein type  $g$ -functions, reads as follows.

**Theorem 1.1.** *Let  $\{T_t\}_{t \geq 0}$  be a symmetric diffusion semigroup. Then for  $1 < q < \infty$ ,  $\alpha > 1/2$ , and  $m \in \mathbb{N}$  with  $m > 2\alpha$ , there is  $\beta := \beta(\alpha) \in \mathbb{R}$  such that*

$$\|f\|_{B_q^\alpha} \asymp \|f\| + \|\mathcal{G}_{m,q,\beta}(f)\|, \quad (5)$$

or equivalently,

$$\|f\|_{B_q^\alpha} \asymp \|f\| + \left( \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m}{\partial t^m} T_t f \right\|^q \frac{dt}{t} \right)^{1/q} \quad (6)$$

with  $m > \beta$ . The result holds for  $q = \infty$  with the modification of the definition.

The definition of Besov spaces in terms of semigroups and modulus of continuity for abstract Hilbert spaces was introduced, for example, by Lion ([21]). There is a large amount of work in the study of Besov spaces and their characterizations in terms of Littlewood-Paley  $g$ -functions for when  $\mathcal{H}$  in  $B_q^\alpha(\mathcal{H})$  is replaced by  $L^p(X)$  space. For example, see [18, 3, 28] for a characterization of  $B_q^\alpha(L^p)$  in terms of the Weierstrass or heat semigroup  $T_t = e^{-Lt}$  (convolution with the heat kernel) associated to a self-adjoint positive definite operator  $L$  on  $L^2$ .

The organization of this paper is as follows: After some historical comments in Section 2, in Section 3 we introduce some notations and definitions, including a short description of our previous results in [22]. In Section 4 we prove few key lemmas and the main result in Theorem 1.1. Finally we apply our results to illustrate Littlewood-Paley-Stein type  $g$ -functions associated to the Poisson and Gauss-Weierstrass semigroups.

## 2. HISTORICAL COMMENTS ON BESOV SPACES

In the classical setting, Besov spaces  $B_q^\alpha(L^p)$  are the set of all functions in  $L^p$  with smoothness degree  $\alpha$  where their (quasi)norm is controlled by  $q$ . These spaces appear in many subfields of analysis and applied mathematics and have two types. The definition of one type uses the Fourier transform (for example see [24, 29]), while the other uses the modulus of continuity or smoothness. The spaces defined by smoothness are more practical in many areas of applied analysis, such as in approximation theory and the decomposition of signals ([7, 8, 6]).

For application purposes, it is natural to decompose a Besov function into simple building blocks and as a result to reduce the study of functions to the study of only the elements in the decomposition. Wavelet and frame theory have played a key role to achieve this goal. In the classical level, this kind of decomposition in terms of “smooth wavelets” using spectral theoretic approach was proved in [12]. A unified characterizations of Besov spaces in terms of atomic decomposition using a group representation theoretic approach was given by Feichtinger and Gröchenig ([9]). New results in this direction in the context of Lie groups and homogeneous manifolds were recently published in [4, 5, 10, 11], and [14]-[17]. For the classification of Besov spaces on compact Riemannian manifolds using continuous and time-frequency localized wavelets with higher vanishing moments we invite the reader to see [15, 16]. For other equivalent definitions of these spaces in terms of abstract wavelets and as interpolation spaces between Hilbert spaces and Sobolev spaces, for example, see the papers [19, 29].

As we mentioned earlier, in this paper we present a description of  $B_q^\alpha(\mathcal{H})$  in term of  $g$ -functions of Littlewood-Paley-Stein type. While the idea behind such an identification is simple, the proofs are very technical and the main difficulties arise when we replace  $L^2$  by any Hilbert space  $\mathcal{H}$ .

## 3. PRELIMINARIES AND DEFINITIONS

Let  $\mathcal{H}$  be a Hilbert space and  $\{T_t\}_{t>0}$  be a semigroup on  $\mathcal{H}$ . We set  $T_0 = I$ . For any  $f \in \mathcal{H}$ , all  $T_t f$  belong to  $\mathcal{H}$  and  $\lim_{t \rightarrow 0^+} T_t f = f$  ([26]). We say  $\{T_t\}_{t \geq 0}$  is *symmetric diffusion* if it satisfies the following conditions.

- (1)  $T_t$  are contractions on  $\mathcal{H}$ , i.e.,  $\|T_t f\| \leq \|f\|$  for all  $f \in \mathcal{H}$ .
- (2)  $T_t$  are symmetric, i.e., each  $T_t$  is self-adjoint on  $\mathcal{H}$ .
- (3)  $T_t$  are positivity preserving, i.e.,  $T_t f \geq 0$  if  $f \geq 0$ .

Symmetric diffusion semigroups occur often in analysis. For examples of this type see [27].

Let  $A$  be the infinitesimal generator of the semigroup  $\{T_t\}_{t \geq 0}$  with domain  $\mathcal{D}(A)$ . By the definition of Besov spaces, the following inclusions hold ([2]):

$$\mathcal{D}(A) \subseteq B_q^\alpha \subseteq \mathcal{H}.$$

$A$  is a densely defined closed linear operator on  $\mathcal{H}$  such that

$$\lim_{t \rightarrow 0^+} \left\| \frac{T_t f - f}{t} - A f \right\| = 0, \quad \forall f \in \mathcal{H}. \quad (7)$$

By the functional calculus,  $T_t f = e^{tA} f$  for all  $f \in \mathcal{H}$  and  $A$  has a representation

$$A = \int_0^\infty \lambda d\mu_\lambda, \quad (8)$$

where  $d\mu_\lambda$  is a projection valued measure ([25]). This implies that for any  $f \in \mathcal{D}(A)$  and  $g \in \mathcal{H}$

$$\langle A f, g \rangle = \int_0^\infty \lambda d(\mu_\lambda f, g). \quad (9)$$

The inner product induces an equivalent definition for the domain of  $A$  given by

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H} : \|A f\|^2 := \int_0^\infty \lambda^2 d(\mu_\lambda f, f) < \infty \right\}.$$

By the density, the inner product (9) also holds for all  $f \in \mathcal{H}$ . As a result of (7) and (8), the operator  $T_t$  has a spectral decomposition

$$T_t = \int_0^\infty e^{-\lambda t} d\mu_\lambda, \quad t > 0.$$

Thus

$$\frac{\partial^m}{\partial t^m} T_t = (-1)^m \int_0^\infty \lambda^m e^{-\lambda t} d\mu_\lambda, \quad \forall m \in \mathbb{N}.$$

We say  $f \in \mathcal{H}$  is a *Paley-Wiener* or *bandlimited* function with respect to the operator  $A$  and the projection valued measure  $\mu_\lambda$  if  $d(\mu_\lambda f, f)$  is supported in the interval  $[a, b]$ ,  $0 < a < b < \infty$ , i.e.,

$$\langle A f, f \rangle = \int_a^b \lambda d(\mu_\lambda f, f).$$

We denote the space of such functions by  $PW_{[a,b]}(A)$  and call it *Paley-Wiener space*. (For an equivalent definition of these spaces using the so-called functional form of the spectral theorem see, for example, [1].) A vector  $f$  in  $\mathcal{H}$  is smooth if it belongs to  $\mathcal{D}(A^k)$ ,  $\forall k \in \mathbb{N}$ . It

is straightforward that  $PW_{[a,b]}(A) \subseteq \cap_{k \in \mathbb{N}} \mathcal{D}(A^k)$ , so every vector in the Paley-Wiener space is smooth.

Let  $\{\psi_j\}_{j \geq 0}$  be a sequence of bounded real-valued functions on  $[0, \infty)$  with  $\text{supp}(\hat{\psi}_j) \subseteq [2^{j-1}, 2^{j+1}]$ . Assume that the following resolution of identity (a.k.a. a discrete version of Calderón decomposition) holds:

$$\sum_{j=0}^{\infty} \hat{\psi}_j(\lambda)^2 = 1. \quad (10)$$

For any  $f \in \mathcal{H}$  and  $j \geq 0$ , define  $f_j := \hat{\psi}_j(A)f$ . Therefore by the resolution of identity and the functional calculus:

$$f = \sum_{j=0}^{\infty} \hat{\psi}_j(A)f_j. \quad (11)$$

Since  $\hat{\psi}_j(A)f \in \cap_{k \in \mathbb{N}} \mathcal{D}(A^k)$ , the functions  $f_j$  are band-limited and smooth. Therefore (11) represents a decomposition of  $f$  in terms of band-limited and smooth functions. In [22] we applied (11) to prove that for  $\alpha > 1/2$  and  $1 < q \leq \infty$

$$\|f\|_{B_q^\alpha} \asymp \|f\| + \left( \sum_{j=0}^{\infty} \left( 2^{j\alpha} \|\hat{\psi}_j(A)f\| \right)^q \right)^{1/q}, \quad \forall f \in \mathcal{H}, \quad (12)$$

provided that both sides are finite. As usual, we use the standard modifications for  $q = \infty$ . We will use the decomposition (11) to prove the main result in Theorem 1.1 in this paper.

*Notation:* By  $\|\cdot\|_{op}$  we shall mean the operator norm, and we will use  $\preceq$  ( $\succeq$ ) when the inequality  $\leq$  ( $\geq$ ) holds up to some uniform constant. Throughout the paper, the equivalence  $\asymp$  indicates  $\preceq$  and  $\succeq$ . By the Besov space (norm) we shall mean the abstract inhomogeneous Besov space (norm).

#### 4. PROOF OF THEOREM 1.1

We need some key lemmas here before we prove our main result. The first one follows.

**Lemma 4.1.** *For any  $j \geq 0$ ,  $1 \leq p < \infty$  and  $k, m \in \mathbb{N}$*

$$\int_0^\infty t^{pk} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \frac{dt}{t} \preceq 2^{-jp(k-m)}. \quad (13)$$

*Proof.* By the functional calculus we have

$$\left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op} = \sup_{\lambda > 0} \left| (-\lambda)^m e^{-\lambda t} \hat{\psi}_j(\lambda) \right| = \sup_{2^{j-1} \leq \lambda \leq 2^{j+1}} |(-\lambda)^m e^{-\lambda t} \hat{\psi}_j(\lambda)| \leq c_1 2^{mj} e^{-2^{j-1}t}.$$

By applying this in (13):

$$\int_0^\infty t^{pk} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \frac{dt}{t} \preceq 2^{jmp} \int_0^\infty t^{pk} e^{-2^{j-1}pt} \frac{dt}{t} = c 2^{-jp(k-m)}.$$

Here, the constants  $c_1$  and  $c$  are independent of  $j$  and  $t$ . This completes the proof of the lemma.  $\square$

**Lemma 4.2.** *Let  $M$  and  $m$  be positive integers such that  $M > 2m$ . Let  $\beta \in \mathbb{R}$ . Then the operator  $K$  defined on  $\mathcal{H}$  by*

$$Kf = \int_0^\infty t^{M-\beta} \left( \frac{\partial^m T_t}{\partial t^m} \right)^2 f \frac{dt}{t}$$

is bounded and for  $1 < q \leq \infty$

$$\|Kf\| \preceq \|\mathcal{G}_{m,q,\beta}(f)\|.$$

*Proof.* By the partition of unity in (10), for any  $f \in \mathcal{H}$  we have

$$Kf = \sum_{j \geq 0} \hat{\psi}_j^2(A) Kf = \sum_{j \geq 0} \hat{\psi}_j(A) Kf_j, \quad (14)$$

where  $f_j = \hat{\psi}_j(A)f$ . Notice that above we used the fact that  $K$  commutes with  $\hat{\psi}_j(A)$ . We continue as follows: First let  $1 < q < \infty$ . Then

$$\begin{aligned} \|Kf_j\| &\leq \int_0^\infty t^{M-\beta} \left\| \left( \frac{\partial^m T_t}{\partial t^m} \right)^2 \hat{\psi}_j(A)f \right\| \frac{dt}{t} \\ &\leq \int_0^\infty t^{M-\beta} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\| \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op} \frac{dt}{t} \end{aligned} \quad (15)$$

$$\leq \left( \int_0^\infty t^{(M-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t} \right)^{1/q} \left( \int_0^\infty t^{(M-m)p} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \frac{dt}{t} \right)^{1/p} \quad (16)$$

$$\preceq \|\mathcal{G}_{m,q,\beta}(f)\| \left( \int_0^\infty t^{(M-m)p} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \frac{dt}{t} \right)^{1/p} \quad (17)$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ . To pass from (15) to (16) we used Hölder's inequality. The inequality (17) holds by the definition of Littlewood-Paley-Stein type  $g$ -functions. By Lemma 4.1, for  $k = M - m$  we can write

$$\int_0^\infty t^{(M-m)p} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \frac{dt}{t} \preceq 2^{-j(M-2m)p}, \quad (18)$$

where the constant in the inequality is independent of  $j$ . By applying this estimation for (17) we arrive at

$$\|Kf_j\| \preceq 2^{-j(M-2m)} \|\mathcal{G}_{m,q,\beta}(f)\|. \quad (19)$$

We will use these estimations for  $Kf_j$  in (14) to complete the proof as follows.

$$\begin{aligned}
\|Kf\| &= \left\| \sum_{j \geq 0} \hat{\psi}_j^2(A) Kf \right\| \\
&\leq \sum_j \|Kf_j\| \|\hat{\psi}_j(A)\|_{op}.
\end{aligned} \tag{20}$$

Due to (10) we have  $\|\hat{\psi}_j(A)\|_{op} \leq 1$ . Therefore, with (19)

$$(20) \preceq \|\mathcal{G}_{m,q,\beta}(f)\| \sum_{j \geq 0} 2^{-j(M-2m)}. \tag{21}$$

With the assumption  $M > 2m$ , the sum is finite. Thus

$$\|Kf\| \preceq \|\mathcal{G}_{m,q,\beta}\|, \tag{22}$$

as desired. For  $q = \infty$ , we use the standard convention for the definition of the norm.  $\square$

**Lemma 4.3.** *Let  $m, j$  be as in the above and  $t, s > 0$ . Then*

$$\left\langle \left( \frac{\partial^m T_t}{\partial t^m} \right)^2 f_j, \left( \frac{\partial^m T_s}{\partial s^m} \right)^2 f_j \right\rangle \geq 2^{4m(j-1)} e^{-2^{j+2}(t+s)} \|f_j\|^2.$$

*Proof.* By the spectral theory and the assumption on the support of  $\hat{\psi}_j$  we have

$$\begin{aligned}
\left\langle \left( \frac{\partial^m T_t}{\partial t^m} \right)^2 f_j, \left( \frac{\partial^m T_s}{\partial s^m} \right)^2 f_j \right\rangle &= \left\langle \int_{\lambda > 0} \lambda^{2m} e^{-2t\lambda} d\mu_\lambda f_j, \int_{w > 0} w^{2m} e^{-2sw} d\mu_w f_j \right\rangle \\
&= \left\langle \int_{\lambda > 0} \lambda^{2m} e^{-2t\lambda} \hat{\psi}_j(\lambda) d\mu_\lambda f, \int_{w > 0} w^{2m} e^{-2sw} \hat{\psi}_j(w) d\mu_w f \right\rangle \\
&= \left\langle \int_{\lambda=2^{j-1}}^{2^{j+1}} \lambda^{2m} e^{-2t\lambda} \hat{\psi}_j(\lambda) d\mu_\lambda f, \int_{w=2^{j-1}}^{2^{j+1}} w^{2m} e^{-2sw} \hat{\psi}_j(w) d\mu_w f \right\rangle \\
&= \int_{\lambda=2^{j-1}}^{2^{j+1}} \lambda^{2m} e^{-2t\lambda} d \left( \int_{w=2^{j-1}}^{2^{j+1}} w^{2m} e^{-2sw} d\langle \mu_\lambda f_j, \mu_w f_j \rangle \right) \\
&\geq 2^{4m(j-1)} e^{-2^{j+2}(t+s)} \int_{\lambda=2^{j-1}}^{2^{j+1}} d \left( \int_{w=2^{j-1}}^{2^{j+1}} d\langle \mu_\lambda f_j, \mu_w f_j \rangle \right). \tag{23}
\end{aligned}$$

Next we aim to show that the integral on the right is  $\|f_j\|^2$ : Notice by the decomposition (8) and the spectral theory we can write

$$\hat{\psi}_j(A)f = \int_{\lambda=0}^{\infty} \hat{\psi}_j(\lambda) d\mu_\lambda f = \int_{\lambda=2^{j-1}}^{2^{j+1}} \hat{\psi}_j(\lambda) d\mu_\lambda f = \int_{\lambda=2^{j-1}}^{2^{j+1}} d\mu_\lambda f_j \quad \forall f \in \mathcal{H}. \tag{24}$$

Therefore, for any  $g \in \mathcal{H}$

$$\langle \hat{\psi}_j(A)f, g \rangle = \int_{\lambda=2^{j-1}}^{2^{j+1}} d\langle \mu_\lambda f_j, g \rangle.$$

Take  $g = f_j = \hat{\psi}_j(A)f$ . The preceding equation with the decomposition (24) implies that

$$\int_{\lambda=2^{j-1}}^{2^{j+1}} d \left( \int_{w=2^{j-1}}^{2^{j+1}} d \langle \mu_\lambda f_j, \mu_w f_j \rangle \right) = \|f_j\|^2. \quad (25)$$

With interfering (25) in (23), the proof holds:

$$\left\langle \left( \frac{\partial^m T_t}{\partial t^m} \right)^2 f_j, \left( \frac{\partial^m T_s}{\partial s^m} \right)^2 f_j \right\rangle \geq 2^{4m(j-1)} e^{-2^{j+2}(t+s)} \|f_j\|^2.$$

□

**Lemma 4.4.** For  $M > 2m$ ,  $M > 2q\alpha - \beta$  and  $f \in \mathcal{H}$

$$\|f_j\| \preceq 2^{-j(4m-M+\beta)/2} \|K f_j\|, \quad (26)$$

where  $f_j = \hat{\psi}(A)f$ ,  $j \geq 0$ , and  $K$  is the operator defined in Lemma 4.2.

*Proof.* By the definition of the operator  $K$  we have

$$\|K f_j\|^2 = \int_{t=0}^{\infty} \int_{s=0}^{\infty} (ts)^{M-\beta} \left\langle \left( \frac{\partial^m T_t}{\partial t^m} \right)^2 f_j, \left( \frac{\partial^m T_s}{\partial s^m} \right)^2 f_j \right\rangle \frac{dt}{t} \frac{ds}{s}. \quad (27)$$

Then (26) is an immediate result of Lemma 4.3 and the following calculations.

$$\begin{aligned} \|K f_j\|^2 &\geq \int_{t=0}^{\infty} \int_{s=0}^{\infty} (ts)^{M-\beta} 2^{4m(j-1)} e^{-2^{j+2}(t+s)} \|f_j\|^2 \frac{dt}{t} \frac{ds}{s} \\ &= 2^{4m(j-1)} \|f_j\|^2 \int_{t=0}^{\infty} \int_{s=0}^{\infty} (ts)^{M-\beta} e^{-2^{j+2}(t+s)} \frac{dt}{t} \frac{ds}{s} \\ &= 2^{4m(j-1)} \|f_j\|^2 \left( \int_{t=0}^{\infty} t^{M-\beta} e^{-2^{j+2}t} \frac{dt}{t} \right)^2 \\ &= 2^{4m(j-1)} 2^{-(j+2)(M-\beta)} \|f_j\|^2 \left( \int_{t=0}^{\infty} t^{M-\beta} e^{-t} \frac{dt}{t} \right)^2 \\ &= c 2^{j(4m-M+\beta)} \|f_j\|^2, \end{aligned}$$

where  $c = 2^{-2(2m+M-\beta)}$ . This completes the proof of the lemma.

□

**Proof of Theorem 1.1.** First we show that for  $1 < q < \infty$

$$\|f\|_{B_\alpha^q} \preceq \|f\| + \left( \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t} \right)^{1/q}.$$

As a result of Lemma 4.4 and the inequality (19) we have

$$2^{j(4m-M+\beta)/2} \|f_j\| \preceq \|K f_j\| \preceq 2^{-j(M-2m)} \left( \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t} \right)^{1/q}.$$



Equivalently,

$$2^{j\alpha q} \|f_j\|^q \preceq 2^{-j/2(M+\beta-2\alpha q)} \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t}.$$

By summing over  $j$  and the assumptions on  $M$  and  $\alpha$ , we arrive to the following inequality:

$$\sum_{j \geq 0} (2^{\alpha j} \|f_j\|)^q \preceq \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t}.$$

Therefore

$$\|f\|_{B_\alpha^q} \preceq \|f\| + \left( \int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t} \right)^{1/q}. \quad (28)$$

We show that the converse of (28) is also true. By the decomposition (10) and an application of Hölder's inequality for  $q, p = \frac{q}{q-1}$ , we have

$$\begin{aligned} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\| &\leq \sum_{j \geq 0} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) f_j \right\| \\ &\leq \sum_{j \geq 0} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op} \|f_j\| \\ &\leq \left( \sum_{j \geq 0} 2^{jq\alpha} \|f_j\|^q \right)^{1/q} \left( \sum_{j \geq 0} 2^{-jp\alpha} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \right)^{1/p} \\ &= \|f\|_{B_q^\alpha} \left( \sum_{j \geq 0} 2^{-jp\alpha} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \right)^{1/p} \end{aligned} \quad (29)$$

Consequently,

$$\int_0^\infty t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t} \leq \|f\|_{B_q^\alpha}^q \int_0^\infty t^{(m-\beta)q} \left( \sum_{j \geq 0} 2^{-jp\alpha} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \right)^{q-1} \frac{dt}{t}. \quad (30)$$

In the rest, we show that the integral on the right hand side of (30) is finite and independent of  $j$ . We shall do this as follows. By the functional calculus

$$\left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op} = t^{-m} \sup_{\lambda} |(\lambda t)^m e^{-\lambda t} \hat{\psi}_j(\lambda)|.$$

Recall that  $|\hat{\psi}_j(\lambda)| \leq 1$  for all  $\lambda$  in the support of  $\hat{\psi}_j$ . This, along with the decay property of  $e^{-x}$ , gives us the following two estimations:

$$\left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op} \preceq t^{-m}, \quad (31)$$

and

$$\left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op} \leq 2^{m(j+1)} e^{-2^{j-1}t}. \quad (32)$$

Let  $\epsilon > 0$ . Take  $\beta \in \mathbb{R}$  such that  $q\beta + 1 > 0$ . Then by applying (31) in (30), for the integral over  $(\epsilon, \infty)$  the following holds:

$$\begin{aligned} \int_{\epsilon}^{\infty} t^{(m-\beta)q} \left( \sum_{j \geq 0} 2^{-jp\alpha} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \right)^{q-1} \frac{dt}{t} &\leq \int_{\epsilon}^{\infty} t^{(m-\beta)q} \left( \sum_{j \geq 0} 2^{-jp\alpha} t^{-mp} \right)^{q-1} \frac{dt}{t} \\ &= \left( \sum_{j \geq 0} 2^{-jp\alpha} \right)^{q-1} \int_{\epsilon}^{\infty} t^{(m-\beta)q} t^{-mq} \frac{dt}{t} \\ &= \left( \sum_{j \geq 0} 2^{-jp\alpha} \right)^{q-1} \int_{\epsilon}^{\infty} t^{-q\beta} \frac{dt}{t} < \infty. \end{aligned} \quad (33)$$

To prove that the integral on the left is also finite on  $(0, \epsilon)$ , we proceed as follows. Pick  $a$  such that  $0 < a \leq m - 1$ . Then for any  $\lambda > 1/2$  we have

$$t^m e^{-\lambda t} \leq (1 + (\lambda t^{-1})^a)^{-1} = \frac{t^a}{\lambda^a + t^a}, \quad 0 < t < \epsilon. \quad (34)$$

Using the inequality (31) once again, we prove that the integral on the right side of (30) is finite on  $(0, \epsilon)$ : Take  $a$  sufficiently large such that  $a > \beta$  and  $\alpha + a > m$ . Then

$$\int_0^{\epsilon} t^{(m-\beta)q} \left( \sum_{j \geq 0} 2^{-jp\alpha} \left\| \frac{\partial^m T_t}{\partial t^m} \hat{\psi}_j(A) \right\|_{op}^p \right)^{q-1} \frac{dt}{t} \quad (35)$$

$$\leq \int_0^{\epsilon} t^{-\beta q} \left( \sum_{j \geq 0} 2^{-jp\alpha} \left( \sup_{2^{j-1} \leq \lambda \leq 2^{j+1}} s^m (1 + (st^{-1})^a)^{-1} \right)^p \right)^{q-1} \frac{dt}{t} \quad (36)$$

$$\begin{aligned} &\leq \int_0^{\epsilon} t^{-\beta q} \left( \sum_{j \geq 0} 2^{-jp\alpha} 2^{jmp} (2^j t^{-1})^{-ap} \right)^{q-1} \frac{dt}{t} \\ &= \left( \sum_{j \geq 0} 2^{-jp(\alpha+a-m)} \right) \int_0^{\epsilon} t^{(a-\beta)q} \frac{dt}{t} < \infty. \end{aligned} \quad (37)$$

Notice that to pass from (35) to (36) we used the estimations (32) and (34), respectively. Consequently, (37) with (33) implies that the integral on right hand side of (30) is finite. Thus

$$\|f\| + \left( \int_0^{\infty} t^{(m-\beta)q} \left\| \frac{\partial^m T_t}{\partial t^m} f \right\|^q \frac{dt}{t} \right)^{1/q} \leq \|f\|_{B_q^\alpha},$$

and we have completed the proof for Theorem 1.1.  $\square$

## 5. APPLICATIONS

We conclude this paper by illustrating our results in two examples.

**Example 5.1.** The Cauchy-Poisson semigroup  $\{P_t\}_{t \geq 0}$  on  $L^2(\mathbb{R}^n)$ , with the convention  $P_0 := I$ , is given by

$$P_t f(x) = \int_{\mathbb{R}^n} f(y) p_t(x - y) dy$$

with the Poisson kernel

$$p_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \quad (x, t) \in \mathbb{R}^n \times (0, \infty).$$

The constant  $c_n$  is chosen such that  $\int p_t(x) dx = 1$ . The Poisson semigroup illustrates a notion of convolution semigroup and  $P_t f = f * p_t$ . Let  $\phi^1$  denote the first derivative of the Poisson kernel for the upper half space at time  $t = 1$ :

$$\phi^1(x) = \frac{\partial}{\partial t} p_t(x)|_{t=1}.$$

The function  $\phi^1$  is integrable with mean value zero, i.e.,  $\int \phi^1(x) dx = 0$ . Similarly, let

$$\phi^m(x) := \frac{\partial^m}{\partial t^m} p_t(x)|_{t=1} \quad m \geq 2.$$

It is easy to show that  $\frac{\partial^m}{\partial t^m} P_t f = f * \phi_t^m$  where  $\phi_t^m(x) = t^{-n} \phi^m(\frac{x}{t})$ ,  $t > 0$ .  $\{P_t\}_{t \geq 0}$  is a symmetric diffusion semigroup (see [27]), and

$$\|f\|_{B_q^\alpha} \asymp \|f\| + \left( \int_0^\infty t^{(m-\alpha)q} \|f * \phi_t^m\|^q \frac{dt}{t} \right)^{1/q}.$$

**Example 5.2.** Let  $\mathcal{H} = L^2(\mathbb{R}^n)$ . The *heat semigroup*  $\{T_t\}_{t \geq 0}$  is defined by the Gauss-Weierstrass formula

$$T_t f(x) = \int_{\mathbb{R}^n} f(y) h_t(x - y) dy, \quad x \in \mathbb{R}^n, \quad t > 0 \quad (38)$$

where  $h_t$  is the heat kernel

$$h_t(x) = c_n t^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad t > 0.$$

We set  $T_0 := I$ . Here,  $c_n = \frac{1}{(4\pi)^{n/2}}$  and  $\int_{\mathbb{R}^n} h_t(x) dx = 1$ . By the definition,  $T_t$ ,  $t > 0$ , is a convolution operator and  $T_t f = f * h_t$ . By Young's inequality  $\|T_t f\| \leq \|f\|$ , and it is easy to verify that  $\{T_t\}_{t \geq 0}$  is a diffusion semigroup. (The semigroup axiom  $T_{t+s} = T_t T_s$  can be obtained by using the Fourier transform.) For the heat semigroup  $\{T_t\}_{t \geq 0}$ , the Besov norm  $\|f\|_{B_q^\alpha}$  is thus equivalent to

$$\|f\| + \left( \int_0^\infty t^{(m-\alpha/2)q} \left\| f * \frac{\partial^m}{\partial t^m} h_t \right\|^q \frac{dt}{t} \right)^{1/q}$$

for any  $m$  with  $2m > \alpha$  and  $\beta = \alpha/2$ .

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